Annals of Operations Research 137, 91–100, 2005

Competition and Cooperation in Non-Centralized Linear Production Games

FRANCISCO RAMÓN FERNÁNDEZ* fernande@us.es
Facultad de Matemáticas, Universidad de Sevilla, 41012 Sevilla, Spain

MARÍA GLORIA FIESTRAS-JANEIRO
Facultade de Económicas, Universidade de Vigo, 36200 Vigo, Spain

IGNACIO GARCÍA-JURADO
Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain

JUSTO PUERTO puerto@us.es
Dep. Estadistica e Investigacion Operative, Facultad de Matemáticas, Universidad de Sevilla, 41012 Sevilla, Spain

Abstract. In this paper we analyze how to improve the benefits of n producers when: (1) each producer i faces a linear production problem given by $\max\{c^i x^i : A^i x^i \leq b^i, x^i \geq 0\}$, and (2) maintaining the production capabilities of each producer is mandatory. In order to maximize the benefits, the producers decide to trade their resources while ensuring their initial individual gains. We study the games which describe this non-centralized linear production situation when players do not cooperate (section two), when players cooperate and side payments are possible (section three), and when players cooperate and side payments are not possible (section four).

Keywords: linear production games, Nash equilibrium, core

1. Introduction

In the classical paper by Owen (1975), the linear programming games are first introduced and analyzed. These games model linear production situations in which a group of players, each of them facing a linear production problem, agree to join their resources and to centralize their production, in order to improve their benefits. This approach implicitly assumes that the producers dismantle their production facilities and transfer their resources to a unique active production plant. Nevertheless, it is easy to imagine that dismantling facilities is not always feasible, mainly because of social side-effects, like unemployment or social disorders. In addition, it is not necessarily optimal. There are cases, see our Example 3.1, where centralizing all the resources is not a good deal. The centralized approach to linear production has been also used to study more general models in papers like Granot (1986) or Curiel, Derks, and Tijs (1989).

* Corresponding author.

In order to search for a compromise between improving the benefits and preserving the production capabilities of the different producers, it is interesting to perform the analysis of non-centralized production models. In Fernández et al. (2003) producers look for a division of their total surpluses, after each of them has implemented a particular optimal production plan. Kalai and Zemel (1982) and Feltkamp et al. (1993) have considered other variations of decentralized linear production situations.

Our approach in this paper looks for an optimal allocation of the resources, improving or at least maintaining the individual optimal benefits of each producer. More explicitly, we consider a situation with n producers of the same goods, with different linear technologies and prices. The producers can trade their resources with the following two constrains: (1) they cannot shut down their production plants, and (2) each production plant should, at worst, maintain its benefits after the trade. Both such constrains aim to avoid negative social side-effects produced by the total or partial dismantling of one or more plants. We propose several approaches to analyzing this situation. The first one (Section 2) assumes that producers behave non-cooperatively. This means that each agent individually requests a portion of the total resources (such that he improves his initial benefits). The goal is to identify those claims leading to Nash equilibria. Also, we give some conditions for the existence of two refinements of the Nash equilibrium concept that we previously introduce. In the second approach (Section 3), producers are supposed to act cooperatively and, moreover, it is assumed that every allocation of the total gains is possible. If a coalition is formed, its members share their resources in order to maximize the sum of their benefits, while improving or, at worst, maintaining the benefits in all their plants. Our purpose, in this second analysis is to identify some core allocation, and to establish the relationship between the payoff of some Nash equilibria of the non-cooperative case with the imputations arising in the cooperative situation. Finally, we analyze the non-transferable utility case (Section 4), where agents cooperate but there are some restrictions for the distribution of the joint benefits. In this context we provide some examples and some results regarding the core of the corresponding NTU game.

2. The non-cooperative analysis

Let us consider a set $N = \{1, \dots, n\}$ of producers. For each agent $i \in N$ we denote by $A^i \in \mathbb{R}^{p \times q}$, $c^i \in \mathbb{R}^q$, and $b^i \in \mathbb{R}^p$ the technology matrix, the unit selling price vector and the resource bundle, respectively. We assume that all the elements of A^i, c^i and b^i are non-negative. Under our hypothesis of linear production technologies, the optimal individual production policy for agent $i \in N$ is obtained solving the problem:

$$\begin{aligned} \max\ & c^i x^i, \\ \text{s.t.: } & A^i x^i \leq b^i, \\ & x^i \geq 0. \end{aligned} \qquad (P^i)$$

For every possible $\bar{b}^i$, we denote $o_i(\bar{b}^i) := \max\{c^i x^i : A^i x^i \leq \bar{b}^i, x^i \geq 0\}$, the value of the linear problem as a function of the right-hand parameters. For any subset $S \subseteq N$ and any profile of vectors $(\bar{b}^1, \ldots, \bar{b}^n)$, we denote $\bar{b}(S) = \sum_{i \in S} \bar{b}^i$.

Assuming that the resources of the different producers are available in a common bundle, their non-cooperative interaction to allocate their resources can be modelled as the following strategic form game. For each agent $i \in N$, its set of pure strategies is given by

$$S^i = \{\varepsilon^i \in \mathbb{R}^p : 0 \leq \varepsilon^i \leq b(N)\}.$$

Player i's payoff function is defined for every profile of strategies $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^n)$ by:

$$K^i(\varepsilon) = \begin{cases} \max\{c^i x^i : A^i x^i \leq \varepsilon^i,\ x^i \geq 0\} & \text{if } \sum_{i \in N} \varepsilon^i \leq b(N), \\ o_i(b^i) & \text{otherwise.} \end{cases}$$

Note first that, in an equilibrium, each agent i will not accept neither dismantling his production facility nor having a benefit smaller than $o_i(b^i)$.

Let us see that this problem is not a trivial one, in the following sense. For any $i \in N$, let x_0^i be an optimal solution of P^i and b_0^i the vector of resources consumed to implement the optimal production plan x_0^i, i.e. $b_0^i = A^i x_0^i$. Then, the profile $(b_0^1, \ldots, b_0^n)$ may not be a Nash equilibrium of the corresponding game. We illustrate it in the following example.

Example 2.1. Let us consider the production system given by $N = \{1, 2\}$, $c^1 = c^2 = 1$, $b^1 = (1, 0)$, $b^2 = (0, 1)$ and

$$A^1 = A^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Next we collect the optimal solutions and their corresponding optimal values and individual consumptions of problems P^1 and P^2.

$$\begin{aligned} x_0^1 = 0,\ \ o_1(b^1) = 0,\ \ b_0^1 = (0, 0), \\ x_0^2 = 0,\ \ o_2(b^2) = 0,\ \ \text{and}\ \ \ b_0^2 = (0, 0). \end{aligned}$$

Obviously, the profile of strategies $(\varepsilon^1, \varepsilon^2) = ((0, 0), (0, 0))$ is not a Nash equilibrium of this game.

In general, the game $(S^1, \ldots, S^n, K^1, \ldots, K^n)$ has many Nash equilibria. For instance, every profile ε in which some players ask for "too much" is a Nash equilibrium in this situation (consider, for instance, that $b(N) > 0$ and take $(\varepsilon^1, \ldots, \varepsilon^n)$ with $|\{i \in N : \varepsilon^i = b(N)\}| \geq 2$). However, these equilibria are not really interesting. Let us look for other Nash equilibria.

It is clear that every ε such that $\sum_{i \in N} \varepsilon^i = b(N)$ and $o_i(\varepsilon^i) \geq o_i(b^i)$, for all $i \in N$, is a Nash equilibrium of this game. In particular, $(b^1, \ldots, b^n)$ is a Nash equilibrium. Notice that there may exist Nash equilibria which do not allocate all the resources. This is illustrated in the next example.

Example 2.2. Let us consider the production system of Example 2.1 after introducing a new restriction. Now, $N = \{1, 2\}$, $c^1 = c^2 = 1$, $b^1 = (1, 0, 1)$, $b^2 = (0, 1, 1)$ and

$$A^1 = A^2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It is clear that the profile $(\varepsilon^1, \varepsilon^2) = ((1/2, 1/2, 1/2), (1/2, 1/2, 1/2))$ is a Nash equilibrium of this game. However, $\varepsilon(N) = (1, 1, 1)$ and $b(N) = (1, 1, 2)$, so there is an excess of the third resource which is not used according to this Nash equilibrium.

The next refinements of the Nash equilibrium concept are interesting in this context.

Definition 2.3. A strategy profile ε is said to be a weakly productive equilibrium if it is a Nash equilibrium and, moreover, there exists $i \in N$ such that $K^i(\varepsilon) > o_i(b^i)$.

Definition 2.4. A strategy profile ε is said to be a productive equilibrium if it is a Nash equilibrium and, moreover, for all $i \in N$, $K^i(\varepsilon) > o_i(b^i)$.

The next proposition provides a necessary and sufficient condition for the non-emptiness of the set of weakly productive equilibria.

Proposition 2.5. The set of weakly productive equilibria is non-empty if and only if there exists $i \in N$ and b_0 such that

$$K^i\left(b_0^{-i}, b(N) - b_0(N \setminus \{i\})\right) > o_i(b^i),$$

where $b_0 = (b_0^1, \ldots, b_0^n)$ is a vector of resources consumed to implement an optimal production plan x_0, i.e. $b_0^i = A^i x_0^i$, and $(b_0^{-i}, b(N) - b_0(N \setminus \{i\}))$ is the profile equal to b_0 in all its components different from the i-th component, which equals $b(N) - b_0(N \setminus \{i\})$.

Proof. It is clear that, if $K^i(b_0^{-i}, b(N) - b_0(N \setminus \{i\})) > o_i(b^i)$ for some $i \in N$, then $(b_0^{-i}, b(N) - b_0(N \setminus \{i\}))$ is a weakly productive equilibrium. Conversely, suppose that ε is a weakly productive equilibrium and take $i \in N$ with $K^i(\varepsilon) > o_i(b^i)$. Then, there exists b_0 such that

$$\varepsilon^i \leq b(N) - b_0(N \setminus \{i\}),$$

and, thus,

$$K^i(b_0^{-i}, b(N) - b_0(N \setminus \{i\})) \geq K^i(\varepsilon) > o_i(b^i).$$

□

It is easy to check that the set of productive equilibria is empty if, for all b_0, vector of resources consumed to implement an optimal production plan, there exists some $i \in N$ such that $K^i(b_0^{-i}, b(N) - b_0(N \setminus \{i\})) = o_i(b^i)$. Besides, a sufficient condition for the non-emptiness of the set of productive equilibria is that there exists such a b_0 for which

$$\varepsilon = \left(\frac{b(N) - b_0(N)}{n} + b_0^1, \ldots, \frac{b(N) - b_0(N)}{n} + b_0^n\right)$$

satisfies that $K^i(\varepsilon) > o_i(b^i)$ for all $i \in N$.

Another interesting property of this model is that the set of payoff undominated Nash equilibria is non-empty. Moreover, our next result characterizes the Nash equilibria whose corresponding payoff vector are undominated.

Proposition 2.6. For any payoff undominated Nash equilibrium $(\bar{\varepsilon}^1, \ldots, \bar{\varepsilon}^n)$ there exists $(\bar{x}^1, \ldots, \bar{x}^n)$ such that $(\bar{\varepsilon}^1, \ldots, \bar{\varepsilon}^n, \bar{x}^1, \ldots, \bar{x}^n)$ is a Pareto-solution of

$$\begin{array}{ll} \max & (c^1x^1, \ldots, c^nx^n) \\ \text{s.t.:} & A^ix^i - \varepsilon^i \leq 0, \quad i \in N, \\ & -c^ix^i \leq -o_i(b^i), \quad i \in N, \\ & \sum_{i\in N} \varepsilon^i \leq b(N), \\ & x^i \geq 0, \varepsilon^i \geq 0, \quad i \in N. \end{array} \qquad \text{(UNP)}$$

Conversely, for any Pareto-solution $(\bar{\varepsilon}^1, \ldots, \bar{\varepsilon}^n, \bar{x}^1, \ldots, \bar{x}^n)$ of problem (UNP) then $(\bar{\varepsilon}^1, \ldots, \bar{\varepsilon}^n)$ is a payoff undominated Nash equilibrium.

The proof is straightforward from the definitions of payoff undominated Nash equilibrium and Pareto-solution of a vector optimisation problem.

3. The transferable utility cooperative analysis

In this section we study our production situation from a cooperative point of view and we suppose that side payments are possible. Thus, this situation can be modelled as the TU-game (N, v) where $v(\emptyset) = 0$ and, for any $S \subseteq N$, $v(S)$ is the maximum of the following linear programming problem

$$\begin{array}{ll} \max & \sum_{i\in S} c^ix^i \\ \text{s.t.:} & A^ix^i - \varepsilon^i \leq 0, i \in S, \\ & -c^ix^i \leq -o_i(b^i), i \in S, \\ & \sum_{i\in S} \varepsilon^i \leq b(S), \\ & x^i \geq 0, \varepsilon^i \geq 0, i \in S. \end{array} \qquad (P^S)$$

In this formulation, players in the coalition S share their resources under the condition that the new distribution has to preserve each individual optimal value and none of the plants are shut-down. In the classical linear production situation, where $c^i = c$, $A^i = A$, for all $i \in N$ (Owen (1975)), it is assumed that the production is centralized by one of the producers and the benefits are later allocated among the producers. The corresponding TU-game (N, w) allocates to every coalition $S \subseteq N$ the maximum value of the linear programming problem

$$\begin{aligned} \max \quad & cz \\ \text{s.t.:} \quad & Az \le b(S) \\ & z \ge 0 \end{aligned} \qquad (P_O^S)$$

The next example illustrates that when the technology matrices and price vectors are different, producers can achieve higher benefits in our model than in the centralized version.

Example 3.1. Let us consider the production system given by $N = \{1, 2\}$, $c^1 = (2, 6, 3)$, $b^1 = (7, 1)$, $c^2 = (2, 4, 6)$, $b^2 = (1, 5)$, and

$$A^1 = A^2 = A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Below we present the optimal solutions and their corresponding optimal values and individual consumptions.

$$\begin{aligned} &x_0^1 = (0, 1, 0), \ \ o_1(b^1) = 6, \ \ b_0^1 = (2, 1), \\ &x_0^2 = (0, 0, 1), \ \ o_2(b^2) = 6, \ \text{ and } \ \ b_0^2 = (1, 3). \end{aligned}$$

In our approach, $v(1, 2)$ is given by the optimal value of the problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{2} c^i x^i \\ \text{s.t.:} \quad & Ax^i - \varepsilon^i \le 0, \quad i = 1, 2, \\ & -c^1 x^1 \le -6, \\ & -c^2 x^2 \le -6, \\ & \varepsilon^1 + \varepsilon^2 \le (8, 6), \\ & x^i \ge 0, \varepsilon^i \ge 0, \quad i = 1, 2. \end{aligned}$$

Its optimal value is 25.8 and an optimal solution is given by

$$(\tilde{x}^1, \tilde{x}^2, \tilde{\varepsilon}^1, \tilde{\varepsilon}^2) = ((0, 3.3, 0), (0, 0.3, 0.8), (6.6, 3.3), (1.4, 2.7)).$$

Notice that the amount of the second resource used by agent two is smaller than the amount he needs to reach his optimal value individually. Now, if we define $w(1, 2) =$

$\max\{\bar{v}^1(1,2), \bar{v}^2(1,2)\}$, where

- $\bar{v}^1(1,2) = \max\{c^1 x \ : \ Ax \leq (8,6), \ x \geq 0\}$,
- $\bar{v}^2(1,2) = \max\{c^2 x \ : \ Ax \leq (8,6), \ x \geq 0\}$,

we get $\bar{v}^1(1,2) = 24$ and $\bar{v}^2(1,2) = 19.2$. Then, $w(1,2) = 24 < v(1,2) = 25.8$.

Although Example 3.1 shows that the centralized procedure (inspired in Owen's model) and our model may give rise to different associated TU-games, we will see now that this is not the case if $c^i = c$, $A^i = A$, for all $i \in N$. Before proving this feature we introduce some notation. For every coalition $S \subseteq N$, we denote by D^S the dual of P^S, which is given by:

$$\begin{array}{ll} \min & -\sum_{i\in S} \delta^i o_i(b^i) + \gamma b(S) \\ \text{s.t.:} & y^i A^i - \delta^i c^i \geq c^i, i \in S, \\ & -y^i + \gamma \geq 0, i \in S, \\ & y^i \geq 0, \delta^i \geq 0, i \in S, \\ & \gamma \geq 0. \end{array} \tag{D^S}$$

The dual problem of P^S_O will be denoted by D^S_O and it is given by:

$$\begin{array}{ll} \min & \alpha b(S) \\ \text{s.t.:} & \alpha A \geq c, \\ & \alpha \geq 0. \end{array} \tag{D^S_O}$$

Theorem 3.2. Let us consider a production situation where $c^i = c$, $A^i = A$, for every $i \in N$. Then, we have $v(S) = w(S)$ for any coalition $S \subseteq N$.

Proof. Let us take any coalition $S \subseteq N$. Clearly, $w(S) \geq v(S)$.
Let z_S denote an optimal solution of problem P^S_O and observe that (x_S, ε_S) is feasible for the problem P^S, where for each $i \in S$

$$(x^i_S, \varepsilon^i_S) = \begin{cases} \left(\dfrac{o_i(b^i) z_S}{\sum_{j\in S} o_j(b^j)}, \dfrac{o_i(b^i) b(S)}{\sum_{j\in S} o_j(b^j)} \right), & \text{if } \sum\limits_{j\in S} o_j(b^j) \neq 0, \\ \left(\dfrac{z_S}{|S|}, \dfrac{b(S)}{|S|} \right), & \text{otherwise.} \end{cases}$$

It follows $v(S) \geq w(S)$.

Thus, we conclude that $v(S) = w(S)$ for any $S \subseteq N$. □

Next we present a result on the core of the TU games associated to non-centralized production situations.

Proposition 3.3. Let $(\bar{x}_N, \bar{\varepsilon}_N)$ be an optimal solution of problem P^N. Let $(\bar{y}_N, \bar{\delta}_N, \bar{\gamma}_N)$ be an optimal solution of problem D^N. Then, $\bar{r} = (c^1\bar{x}^1, \ldots, c^n\bar{x}^n)$ is an imputation of the game (N, v). Moreover, the vector

$$\hat{r} = \left(\bar{\gamma}_N b^1 - \bar{\delta}_N^1 o_1(b^1), \ldots, \bar{\gamma}_N b^n - \bar{\delta}_N^n o_n(b^n)\right)$$

belongs to the core of the game (N, v).

Proof. It is clear that $\bar{r}$ is an imputation of (N, v). To check that $\hat{r}$ belongs to the core of (N, v) it is enough to notice that

$$\sum_{i \in N} \left(\bar{\gamma}_N b^i - \bar{\delta}_N^i o_i(b^i)\right) = \bar{\gamma}_N b(N) - \sum_{i=1}^{n} \bar{\delta}_N^i o_i(b^i) = v(N)$$

and that the restriction of $(\bar{y}_N, \bar{\delta}_N, \bar{\gamma}_N)$ to S is a feasible solution of D^S for any coalition $S \subseteq N$. □

To finish this section, we present two remarks.

Remark 3.4. If $(\bar{x}, \bar{\varepsilon})$ is an optimal solution of P^N then $\bar{\varepsilon}$ is a payoff undominated Nash equilibrium of the non-cooperative game defined in Section 2, where $K^i(\bar{\varepsilon}) = c^i\bar{x}^i$, for all $i \in N$. Moreover, $\bar{\varepsilon}$ is a weakly productive equilibrium unless $v(N) = \sum_{i \in N} o_i(b^i)$.

Remark 3.5. Notice that the game (N, v) defined in this section is non-negative and totally balanced. Since the class of non-negative totally balanced games coincides with the class of linear production games, there exists a linear production game that induces the game (N, v). In this case the construction can be done in terms of the elements of the production system (technologies, prices and resources.)

4. The non-transferable utility cooperative analysis

In this section we consider the production situation from a cooperative point of view, but assuming that side payments are not possible. Therefore, this situation should be modelled as an NTU game, more precisely, an NTU market game (see, for instance, Owen (1995) for details on market games). If a coalition S is to form, players can redistribute their resources ensuring their individual gains, i.e., they can obtain any tuple $(\varepsilon^i)_{i \in S}$ such that:

- $\sum_{i \in S} \varepsilon^i = b(S)$,
- $\varepsilon^i \geq 0$, for all $i \in S$,
- $o_i(\varepsilon^i) \geq o_i(b^i)$, for all $i \in S$.

Any such tuple is called a feasible allocation for S. Then, for every S, $V(S)$ is the set of all $z \in \mathbb{R}^n$ for which there exists $(\varepsilon^i)_{i\in S}$, a feasible allocation for S, such that $z_i \leq o_i(\varepsilon^i)$.

It is immediate to check that, defined in this way, the functions o_i are concave (just note that $o_i(\varepsilon^i) = \min\{\varepsilon^i y^i : y^i A^i \geq c^i, y^i \geq 0\}$), continuous and non-decreasing, so the core of (N, V) is non-empty, a competitive equilibrium exists and it provides a core allocation.

An interesting subset of $core(N, V)$ is the following set, that will be called $subcore(N, V)$:

$$\{z \in PB(V(N)) : (z_i)_{i\in S} \geq (y_i)_{i\in S}, \text{ for all } y \in V(S) \text{ and all } S \subset N\},$$

where $PB(V(N))$ denotes the Pareto boundary of $V(N)$. Next below, we see that there exist production systems where the *subcore* of the associated NTU game defined as above is non-empty, and production systems where it is empty.

Example 4.1. Let $N = \{1, 2, 3\}$, $c = (4, 6, 8)$, $b^1 = (6, 0)$, $b^2 = (2, 5)$, $b^3 = (1, 5)$, and $A = \left(\begin{smallmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \end{smallmatrix}\right)$ be a production system. Below we collect the optimal solution sets, optimal values, and individual consumption of resources of problems P^i for all $i = 1, 2, 3$.

$$\begin{array}{lll} x_0^1 = (0, 0, 0), & o_1(b^1) = 0, & b_0^1 = (0, 0), \\ x_0^2 \in conv\{(1/2, 0, 3/2), (0, 1/5, 8/5)\}, & o_2(b^2) = 14, & b_0^2 = (2, 5), \\ x_0^3 = (0, 0, 1), & o_3(b^3) = 8, & \text{and} \quad b_0^3 = (1, 3). \end{array}$$

Let us denote by $comph\{A\}$ the comprehensive hull of the set A. The associated NTU game is defined by $V(1) = comph\{0\}$, $V(2) = comph\{14\}$, $V(3) = comph\{8\}$, $V(1, 2) = comph\{(12, 14), (0, 26)\}$, $V(1, 3) = comph\{(16, 8), (0, 24)\}$, $V(2, 3) = comph\{(16, 8), (14, 10)\}$, and $V(1, 2, 3) = comph\{E\}$, where

$$E = \{(0, 14, 24), (0, 30, 8), (16, 14, 8)\}.$$

Let us consider the production situation given agents 1 and 2 described above and let $(\{1, 2\}, \bar{V})$ the associated NTU game. In this system, the allocation $(0, 26)$ belongs to $subcore(\{1, 2\}, \bar{V})$. But, when all three agents are considered, it holds that $subcore(N, V)$ is empty. This comes from the fact that $(0, 26, 24)$ does not belong to the Pareto boundary of $V(N)$.

In the sequel we provide a characterization of the non-emptiness of $subcore(N, V)$. For any $\hat{z} \in V(N)$, we define the family of TU-games $\{(N, v_{\hat{z}}^j), j \in N\}$ such that $v_{\hat{z}}^j(N) = \hat{z}_j$ and, for any non-empty $S \subset N$,

$$v_{\hat{z}}^j(S) = \begin{cases} \max\limits_{(\varepsilon^i)_{i\in S} \text{ feasible}} o_j(\varepsilon^j) & \text{if } j \in S, \\ 0 & \text{if } j \notin S. \end{cases}$$

Theorem 4.2. It holds that $subcore(N, V) \neq \emptyset$ if and only if there exists $\hat{z} \in V(N)$ such that all the TU-games $(N, v^j_{\hat{z}})$, $j \in N$, have a non-empty core.

Proof. Assume that all the games $(N, v^j_{\hat{z}})$, $j \in N$, have a non-empty core (for some $\hat{z} \in V(N)$). Then, for every $j \in N$, there exists $z^j = (z^j_1, \ldots, z^j_n)$ a core element of $(N, v^j_{\hat{z}})$. Hence, for any $S \subset N$ and any $j \in S$,

$$\hat{z}_j = \sum_{i \in N} z^j_i \geq \sum_{i \in S} z^j_i \geq v^j_{\hat{z}}(S) \geq y_j,$$

for all $y \in V(S)$. Thus, there must exist $z \in subcore(N, V)$ with $z \geq \hat{z}$. Conversely, take $\hat{z} \in subcore(N, V)$ and define $\bar{z}^j = (0, \ldots, \hat{z}^j_j, \ldots, 0)$. Clearly $\bar{z}^j \in core(N, v^j_{\hat{z}})$. □

Acknowledgments

The authors thank Spanish *Ministry of Science and Technology*, FEDER, and *Xunta de Galicia* for partially supporting their research through grants BFM2001-2378, BFM2004-0909, BEC2002-04102-C02-02, and PGIDT03PXIC20701PN. The valuable comments of two anonymous referees are gratefully acknowledged.

References

Curiel, I., J. Derks, and S. Tijs. (1989). "On Balanced Games and Games with Committee Control." *OR Spektrum* 11, 83–88.

Feltkamp, V., A. van den Nouweland, P. Borm, M. Koster, and S. Tijs. (1993). "Linear Production with Transport of Products, Resources, and Technology." *ZOR* 38, 153–162.

Fernández, F.R., M.G. Fiestras, I. García-Jurado, and J. Puerto. (2003). "On the Allocation of Excesses of Resources in Linear Production Problems." In U. Leopold-Wildburger, F. Rendl, and G. Wäscher (eds.), *Operations Research Proceedings 2002.* Heidelberg: Springer-Verlag, pp. 465–470.

Granot, D. (1986). "A Generalized Linear Production Model: A Unifying Model." *Mathematical Programming* 34, 212–222.

Kalai, E. and E. Zemel (1982). "Generalized Network Problems Yielding Totally Balanced Games." *Operations Research* 30, 998–1008.

Owen, G. (1975). "The Core of Linear Production Games." *Mathematical Programming* 9, 358–370.

Owen, G. (1995). *Game Theory*. San Diego: Academic Press.